

Fixed Point Theorems in Hausdorff Topological Vector Spaces and Economic Equilibrium Theory

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October, 2003

Abstract

The aim of this paper is to develop fixed point theorems in Hausdorff topological vector spaces that are suitable for the purpose of economic equilibrium theory. The special concept we have used here is the “direction structure” that characterizes mappings in the economic theory, (preferences, excess demands, and the like,) adequately, and enables us to modify problems on mappings into those on a structure of the base set. Especially, since our mathematical generalization may directly be related to the continuity and/or convexity of individual preferences, we may obtain existence theorems of maximal points, Pareto optimal allocations, and price equilibria for Gale-Nikaido-Debreu abstract economies under quite natural conditions.

Keywords : Fixed point theorem, Non-ordered Preference, Direction structure, Gale-Nikaido-Debreu theorem, Market equilibrium.

JEL classification: C62; D50

1 INTRODUCTION

In the economic theory, fixed point theorem is one of the most important mathematical tools that enable us to construct the concept of economic equilibria. Under the earliest formulation as in Arrow and Debreu (1954), Nikaido (1956), etc., the economic equilibrium was treated as a fixed point of a continuous mapping constructed by continuous excess demands and price formation functions. The problem was lately reformulated as a general coincidence property for restriction and preference correspondences including cases with non-ordered preferences (c.f. Shafer and H.F.Sonnenschein (1975), Gale and Mas-Colell (1975)).

In this paper, we prove fixed point theorems and theorems on economic equilibria under weak conditions on local directions of mappings in Hausdorff topological vector spaces. Our generalization directly aims to support a weak condition on the convexity and continuity of preference correspondences for the existence of economic equilibria.

Results in this paper are based on recent researches in Urai (2000), Urai and Hayashi (2000), and Urai and Yoshimachi (2002) on fixed point theorems for multi-valued mappings and economic equilibria in Hausdorff topological vector spaces. We have developed here a way to generalize

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them by using more essential mathematical structure defining a notion of “directions” in topological vector spaces. The notion enables us to characterize mappings in economics (especially, preference mappings,) adequately, and to modify problems on properties of such mappings into problems on properties of spaces. Continuity for the direction of set valued mappings may be reduced into topological features for the direction structure of spaces, and the generalization of continuity condition for the existence of fixed points as in Urai (2000) may be reformulated here as a generalization of conditions for a subset of vector spaces on which all continuous functions have fixed points.

In this paper, we apply these results to general existence theorems of maximal points, Pareto optimal allocations, and price equilibria for an abstract economy of the Gale-Nikaido-Debreu type. The result may also be applied to the existence of equilibrium for an abstract economy of the Arrow-Debreu type with (possibly) non-ordered, non-convex, and/or non-continuous preferences and constraint correspondences in Hausdorff topological vector spaces which may not necessarily be locally convex.¹

2 FIXED POINT THEOREMS

In this section, we define a structure on a topological vector space which generalizes some concepts included in our ordinary usage of the word a “direction” for a vector, and we show a fixed point theorem depending merely on local conditions for such directions associated with a mapping without using the concept of continuity and/or convexity. Moreover, the structure is also used to describe a weaker condition on a subset of vector spaces under which every continuous function has a fixed point.

Let E be a Hausdorff topological vector space over the real field R , and let X be a subset of E . We define a structure which represents the set of points in *direction* $y - x$ at x in X for each $x, y \in X$ as follows. For each pair (x, y) of an element of $X \times X$, define a subset $V(x, y)$ of X satisfying:

$$(A0) \forall x, y \in X, V(x, y) \text{ is a convex subset of } X.$$

$$(A1) \forall x, y \in X, x \notin V(x, y),$$

$$(A2) \forall x, y, z \in X, (z \in V(x, y)) \rightarrow (y \in V(x, z)).^2$$

We call the set $V(x, y) \subset X$ the *set of points in direction* $y - x$ at x , or simply, the *set of direction* y from x , and we say that X has a *direction structure* V .³

An example for such a structure is obtained by using the inner product. When the inner product is defined on E , we may define *the inner product direction structure*, V on $X \subset E$, as $V(x, y) = \{z \in X | \langle y - x, z - x \rangle > 0\}$.⁴ For a vector space E with an algebraic dual E^* , we may also define V on $X \subset E$ as $V(x, y) = \{z \in X | p(x, y)(z - x) > 0 \wedge p(x, z)(y - x) > 0\}$ whenever $p : X \times X \rightarrow E^*$ is a mapping such that $p(x, x) = 0$ for all $x \in X$. Another example may be obtained when there is a correspondence $\varphi : X \rightarrow X$ such that $x \notin \text{co } \varphi(x)$ for all $x \in X \setminus K$, where $K = \{x \in X | x \in \varphi(x)\}$ is the *fixed point set* of φ .⁵ In this case, we may define direction structure V_φ on $X \subset E$ as

¹See also Urai-Yoshimachi(2002) for a similar result based on Urai-Hayashi (2000), where a direction of mapping is treated as an element of the dual space of a locally convex base space.

²Condition (A2) may be replaced with a weaker condition (A2') $\forall x, y, z \in X, (w \in V(x, y) \cap V(x, z)) \rightarrow (\forall \lambda \in [0, 1], w \in V(x, \lambda y + (1 - \lambda)z))$. Note that under (A1) and (A2), $V(x, x) = \emptyset$ for all $x \in X$.

³The similar notion has already been treated in Urai (2000) [Concluding Remarks].

⁴Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ the vector space duality operation including the inner product.

⁵The notation $\text{co } A$ denotes the convex hull of A .

$V_\varphi(x, y) = X \cap \text{co } \varphi(x)$, if $x \in X$ and $y \in \text{co } \varphi(x)$, else $V_\varphi(x, y) = \emptyset$. We call direction structure V_φ the *direction structure induced by φ* on X .

Suppose that X has a direction structure, $V : X \times X \rightarrow X$. We say that a correspondence (possibly empty valued) $\varphi : X \rightarrow X$ has a *locally common direction y^x at x* (under V) if there exists an open neighbourhood $U(x)$ of x such that $\varphi(z) \subset V(z, y^x)$ for all $z \in U(x)$. Based on the direction structure, we have the following fixed point theorem.

THEOREM 1 : (Fixed Point Theorem for Mappings Having Locally Common Directions) *Let X be a non-empty compact convex subset of Hausdorff topological vector space E having direction structure V , and let $\varphi : X \rightarrow X$ be a non-empty valued correspondence. Suppose that φ has a locally common direction under V at every x such that $x \notin \varphi(x)$. Then, φ has a fixed point.*

PROOF : Assume that φ has no fixed point. Then, φ has a locally common direction at each $x \in X$. Since X is compact, we have points $x^1, \dots, x^n \in X$, open neighbourhoods $U(x^1), \dots, U(x^n)$ of each x^1, \dots, x^n in X such that $\bigcup_{t=1}^n U(x^t) \supset X$, together with points $y^{x^1}, \dots, y^{x^n} \in X$ satisfying for each $t = 1, \dots, n$, $\varphi(z) \subset V(z, y^{x^t})$ for all $z \in U(x^t)$. Let $\beta_t : X \rightarrow [0, 1]$, $t = 1, \dots, n$, be a partition of unity subordinated to $U(x^1), \dots, U(x^n)$. Let us define a function f on $D = \text{co } \{y^{x^1}, \dots, y^{x^n}\}$ to itself as $f(x) = \sum_{t=1}^n \beta_t(x)y^{x^t}$. Then, f is a continuous function on the finite dimensional compact set D to itself. Hence, f has a fixed point $z^* = f(z^*) = \sum_{t=1}^n \beta_t(z^*)y^{x^t}$ by Brouwer's fixed point theorem. On the other hand, for all t such that $z^* \in U(x^t)$, (i.e., $\beta_t(z^*) > 0$), $\varphi(z^*) \subset V(z^*, y^{x^t})$, so that (by condition (A2)) for an element $y^* \in \varphi(z^*)$ arbitrarily fixed, we have $y^{x^t} \in V(z^*, y^*)$ for all t such that $\beta_t(z^*) > 0$. Since $V(z^*, y^*)$ is convex, we have $z^* = \sum_{t=1}^n \beta_t(z^*)y^{x^t} \in V(z^*, y^*)$, which contradicts the fact $z^* \notin V(z^*, y^*)$ under condition (A1). ■

Every non-empty convex valued correspondence having open lower sections, $\varphi : X \rightarrow X$, has a locally common direction at each x such that $x \notin \varphi(x)$ under the direction structure induced by φ . Hence, by considering the induced direction structure, the above theorem may be considered as an extension of Browder's fixed point theorem (c.f., Browder (1968)). The theorem also includes one of the main fixed point theorems in Urai (2000) (Theorem 1 (K^*)). In the following, by using the concept of direction structure, we further extend the result to more general cases with mappings having locally "continuous" directions.⁶

We say that a direction structure, $V : X \times X \rightarrow X$, is *lower topological* on a certain subset $A \subset X \times X$ if the following (A3) is satisfied.⁷

(A3) For all $(x, y) \in A$, $V(x, y) \neq \emptyset$ implies that $\exists W(x, y)$, an open neighbourhood of $(x, y) \in A \subset X \times X$ such that $\bigcap_{(z, w) \in W(x, y)} V(z, w) \neq \emptyset$.

A correspondence $\varphi : X \rightarrow X$ is said to *have a locally continuous direction at x* (under a structure V) if there exists an open neighbourhood $U(x)$ and a continuous function $y : U(x) \rightarrow X$ such that $\varphi(z) \subset V(z, y(z))$ for all $z \in U(x)$.

⁶The difference is important especially when values of mappings are closed, e.g., if $f : X \rightarrow X$ is single valued, under the induced direction structure, f fails to have a locally common direction as long as it is not locally constant, though f always has a continuous direction as long as it is continuous.

⁷Indeed, the following is nothing but a condition for the lower section of the correspondence V , i.e., for each $V(x, y) \neq \emptyset$ there is at least one element whose lower section is a neighbourhood of (x, y) .

THEOREM 2 : (Fixed Point Theorem for Mappings Having Locally Continuous Directions I) *Let X be a non-empty compact convex subset of Hausdorff topological vector space E having lower topological direction structure V on $X \times X$, and let $\varphi : X \rightarrow X$ be a non-empty valued correspondence. Suppose that φ has a locally continuous direction (under V) at every x such that $x \notin \varphi(x)$. Then, φ has a fixed point.*

PROOF : Assume that φ has no fixed point. Then, φ has a locally continuous direction $y^x : U(x) \rightarrow X$, $\varphi(z) \subset V(z, y^x(z))$ for all $z \in U(x)$ at each $x \in X$. Since X is compact, we have finite points $x^1, \dots, x^n \in X$, open neighbourhoods $U(x^1), \dots, U(x^n)$ of each x^1, \dots, x^n in X such that $\bigcup_{t=1}^n U(x^t) \supset X$, together with continuous functions $y^{x^1} : U(x^1) \rightarrow X, \dots, y^{x^n} : U(x^n) \rightarrow X$ satisfying for each $t = 1, \dots, n$, $\varphi(z) \subset V(z, y^{x^t}(z))$ for all $z \in U(x^t)$. Let $\beta_t : X \rightarrow [0, 1]$, $t = 1, \dots, n$, be a partition of unity subordinated to $U(x^1), \dots, U(x^n)$. Let us define a function f on X to itself as $f(x) = \sum_{t=1}^n \beta_t(x) y^{x^t}(x)$, where $y^{x^t}(x)$ denotes 0 for each $x \notin U(x^t)$. Then, f is a continuous function on X to itself. Since $\emptyset \neq \varphi(z) \subset V(z, \sum_{t=1}^n \beta_t(z) y^{x^t}(z))$ for all $z \in X$, by defining $\Phi(z)$ as $\Phi(z) = V(z, f(z))$, the mapping $\Phi : X \rightarrow X$ is non-empty valued and $z \notin \Phi(z)$ for all $z \in X$. Moreover, by (A3) and by the continuity of f , Φ has a locally common direction at each $x \in X$. Hence, by Theorem 1, Φ has a fixed point, so that we have a contradiction. ■

When the topology on a vector space is given by the inner product, the inner product direction structure, $V(x, y) = \{z | \langle z - x, y - x \rangle > 0\}$, is lower topological. Hence, the theorem gives a sufficient generality for finite dimensional cases under the inner product direction structure. The following corollary which is an immediate consequence of the above argument shows one of the most interesting (and important) features of our theorem. Note that no continuity and no convexity are assumed on mapping P except for those on the function, d .⁸

COROLLARY 2.1 : *Let X be a non-empty compact convex subset of R^n and let $P : X \rightarrow X$ be a correspondence having continuous direction $d : X \rightarrow X$ under the Euclidean inner product direction structure, i.e., $D(x) \cdot (y - x) > 0$ for all $y \in P(x)$, where $D(x) = d(x) - x$ for each $x \in X$. Then, there is a point x such that $P(x) = \emptyset$.⁹*

PROOF : Note that P has no fixed point since $D(x) \cdot (y - x) > 0$ for all $y \in P(x)$ for all $x \in X$. If P is non-empty valued, however, under the inner product direction structure, P has a fixed point x^* by Theorem 2. ■

A slight modification of the above theorem into the case with single valued mapping will be useful in the later. (Note that the next corollary is trivial when the topology on space E is locally convex by the fixed point theorem of Schauder-Tychonoff.)

COROLLARY 2.2 : (Fixed Point Theorem for a Continuous Mapping I) *Let X be a non-empty compact convex subset of Hausdorff topological vector space E and let $f : X \rightarrow X$ be a continuous function. Suppose that there is a direction structure V on X such that $f(x) \in V(x, f(x))$ at each $x \neq f(x)$ and V is lower topological on the graph of f . Then, f has a fixed point.*

⁸For the Euclidean inner product in R^n , we write $x \cdot y$ instead of $\langle x, y \rangle$.

⁹In view of economics, above $D(x)$ may be interpreted as a generalized concept for the continuous first derivative of a utility function.

PROOF : Assume the contrary. Then, the mapping $\Phi(x) = V(x, f(x))$ on X to X is non-empty convex valued. Moreover, Φ has a locally common direction at each $x \in X$ since f is continuous and (A3) is satisfied at each $(x, f(x))$. Hence, Φ has a fixed point by Theorem 1, so that we have a contradiction. ■

By reading the proof of Theorem 2 or Corollary 2.2, one can see that the lower topological property, (A3), is nothing but a sufficient condition for cases with locally “continuous” directions to be reduced to cases with locally “common” directions in Theorem 1. Unfortunately, however, except for the above inner product cases, there seems to be no obvious way to define a direction structure which is lower topological on $X \times X$. Even for cases such that E together with the topological dual E' forms the duality, and $V(x, y) = \{z \in X | \langle p(x, y), z - x \rangle > 0\}$, where p is a function on $X \times X$ to E' , defines a direction structure, we should induce a compact convergence topology on E' since for V to be lower topological, we have to assure the duality operation, $\langle \cdot, \cdot \rangle$ to be jointly continuous.¹⁰

For fixed point arguments in more general topological spaces, it is more desirable to use the following alternative condition, (A4), to (A3).¹¹ We say that a direction structure, $V : X \times X \rightarrow X$, is *upper topological* on a certain subset $A \subset X \times X$ if the following (A4) is satisfied.¹²

(A4) For all $(x, y) \in A$, if $V(x, y) \neq \emptyset$, then there are two neighbourhoods U_x of x and U'_y of y in X such that $U_x \cap \text{co}U'_y = \emptyset$.

See Figure 1. We are considering the case that the space E may not be locally convex, so that there

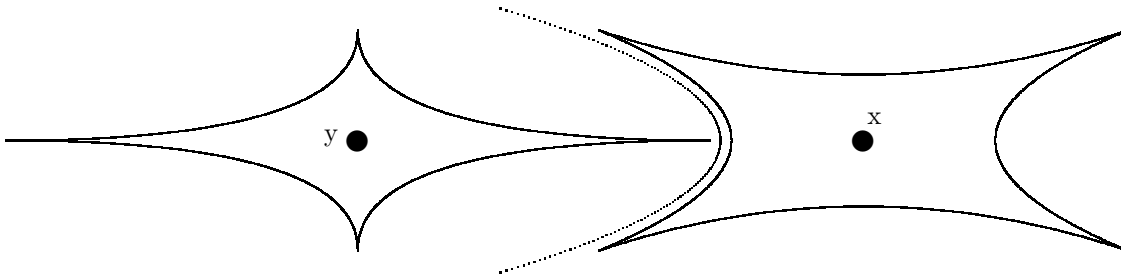


Figure 1: Condition (A4)

may not exist a convex neighbourhood base at each point. Condition (A4) has its meaning only for cases with $V(x, y) \neq \emptyset$. If we do not allow $V(x, y)$ to have empty value as long as $x \neq y$, then a typical condition assuring (A4) is that for each $x, y, x \neq y, V(x, y)$ includes at least one neighbourhood of y . Hence, (A4) is typically a condition for the upper section of the correspondence, V .¹³ Now we have the following theorem.

¹⁰Under the compact convergence topology on E' , however, we further generalize our results to the case with mappings having *compact valued upper semicontinuous directions*. See, Urai-Yoshimachi (2002).

¹¹Of course, it is not saying that condition (A4) is more general than condition (A3). Indeed, in the proof of next theorem, (A4) is used as a sufficient condition for the space to have a certain lower topological direction structure.

¹²As stated below, the condition is closely related to a property for the upper section of the correspondence, V .

¹³In such a case, by considering $V(z, x)$ and $V(z, y)$ for $z = (x + y)/2$, we have two disjoint convex neighbourhoods of x and y for each $x \neq y$. Note, however, that the whole space, E , may not be locally convex even in such cases. For an example, consider ℓ^p for $0 < p < 1$.

THEOREM 3 : (Fixed Point Theorem for Mappings Having Locally Continuous Directions II) *Let X be a non-empty compact convex subset of Hausdorff topological vector space E having upper topological direction structure V on $X \times X$, and let $\varphi : X \rightarrow X$ be a non-empty valued correspondence. Suppose that φ has a locally continuous direction (under V) at every x such that $x \notin \varphi(x)$. Then, φ has a fixed point.*

PROOF : As in the proof of Theorem 2, assume that φ has no fixed point, and obtain the continuous function $f : X \rightarrow X$. Note that the non-emptiness of $V(x, f(x))$ for all $x \in X$ means that f has no fixed point. Let $A \subset X \times X$ be the graph of f . On A , we modify the direction structure V into V' as follows. For each $(x, f(x)) \in A$, $V(x, f(x)) \neq \emptyset$ implies (under (A4)) that there are two disjoint neighbourhoods, U_x and $U'_{f(x)}$, of x and $f(x)$ such that $U_x \cap \text{co} U'_{f(x)} = \emptyset$. Hence, by the continuity of f , there is an open neighbourhood $O_x \subset U_x$ of x such that for all $z \in O_x$, $f(z) \in U'_{f(x)}$ and $O_x \cap \text{co} U'_{f(x)} = \emptyset$. Moreover, since for a vector space topology it is always possible to take a closed neighbourhood base, O_x may be chosen so that there are two neighbourhoods of $f(x)$ in X , $U^1_{f(x)}$, $C^1_{f(x)}$, satisfying:

- (1) $\text{int } U'_{f(x)} \supset U^1_{f(x)} \supset C^1_{f(x)}$,¹⁴
- (2) $U^1_{f(x)}$ is open and $C^1_{f(x)}$ is closed (i.e., compact) in X ,
- (3) $\forall z \in O_x, z + (U'_{f(x)} - x) \supset U^1_{f(x)}$,
- (4) $\forall z \in O_x, f(z) \in C^1_{f(x)}$,
- (5) O_x is closed in X .

Indeed, by the property of vector space topology, we may chose a circled 0 neighbourhood U in E so that $(U+U) \cap X \subset U'_{f(x)} - f(x)$.¹⁵ Then, by taking O_x so small that $O_x \subset x+U$, we have for all $z \in O_x \subset x+U$, $((x-z)+U) \cap X \subset (U+U) \cap X \subset U'_{f(x)} - f(x)$, i.e., $(f(x)+U) \cap X \subset z+(U'_{f(x)} - x)$. Hence, by setting $U^1_{f(x)} = (f(x) + \text{int } U) \cap X$ and choosing $O_x \subset (x + \text{int } U) \cap X$, (1),(2),(3) are satisfied. For conditions (4) and (5), chose $C^1_{f(x)} \subset U^1_{f(x)}$ as an arbitrary closed neighbourhood of $f(x)$ and redefine a closed O_x (smaller than before) by considering the continuity of f . On each O_x , define $V^x(z, f(z))$ as $(z + (\text{int } (\text{co} U'_{f(x)} - x)) \cap X) \supset U^1_{f(x)} \supset C^1_{f(x)} \supset f(O_x)$. Since X is compact, there are finite points x^1, \dots, x^m such that O_{x^1}, \dots, O_{x^m} covers X . For each $(x, f(x)) \in A$, let

$$V'(x, f(x)) = \bigcap_{t \in I(x)} V^{x^t}(x, f(x)),$$

where $I(x) \neq \emptyset$ denotes the subset of $\{1, 2, \dots, m\}$ such that $(t \in I(x)) \iff (x \in O_{x^t})$. $V'(x, f(x))$ is non-empty since $(t \in I(x)) \iff (x \in O_{x^t}) \implies (f(x) \in C^1_{f(x^t)})$. Moreover, by defining $V'(x, y) = \emptyset$ for all $y \notin V'(x, f(x))$, V' is a direction structure which is lower topological on $X \times X$. Indeed, for each $(z, f(z))$, let I_z be the set of index t such that $z \in O_{x^t}$. Then, by defining W^1 as a neighbourhood of z in X such that W^1 does not intersect O_{x^t} for all $t \notin I_z$, and W^2 as $\bigcap_{t \in I_z} U^1_{f(x^t)}$, W^1 and W^2 are open neighbourhoods of z and $f(z)$, respectively, and for $W = W^1 \times W^2$, $\bigcap_{(x,y) \in W} V'(x, y) \ni f(z)$. This, together with the openness for the value of V' ,

¹⁴Notation $\text{int } A$ denotes the interior of A . In this proof, all interiors are taken with respect to the topology on the whole space, E .

¹⁵See Schaefer (1971) [p.14, Theorem 1.2].

implies that V' satisfies condition (A3). Hence, f has a fixed point by Corollary 2.2. Since f has no fixed point, we have a contradiction. ■

An important corollary to the above theorem is the case that the mapping φ is a continuous function $f : X \rightarrow X$. In this case, we are able to rewrite the condition with more familiar concepts without an essential loss in generality.

COROLLARY 3.1 : (Fixed Point Theorem for a Continuous Mapping II) *Let X be a non-empty compact convex subset of Hausdorff topological vector space E such that*

(T) every two different points, $x \neq y$, in E has two disjoint neighbourhoods, $U_x \cap U_y = \emptyset$, at least one of which is convex.

Then, every continuous function, $f : X \rightarrow X$, has a fixed point.

PROOF : Note that in this proof, condition (T) is used merely for pairs of points on the graph of f , i.e., x and $f(x)$ such that $x \neq f(x)$. Assume that f has no fixed point. Then, for each $(x, f(x)) \in X \times X$, x and $f(x)$, respectively, have neighbourhoods, U_x and $U_{f(x)}$, such that $U_x \cap U_{f(x)} = \emptyset$ and at least one of which is convex. Since the topology of vector space is translation invariant, we may suppose $U_{f(x)}$ is convex without loss of generality. Hence, by defining $V(x, f(x)) = U_{f(x)}$ for each $x \in X$ and $V(x, y) = \emptyset$ for each x and $y \notin U_{f(x)}$, we obtain a direction structure which is upper topological on $X \times X$. Since f has a continuous direction under V at everywhere, f has a fixed point by Theorem 3, so that we have a contradiction. ■

The above condition, (T), is automatically satisfied when the topology on E is locally convex. The converse is not true, i.e., there is a topological vector space whose topology is not locally convex but satisfies condition (T).¹⁶ Hence, Corollary 3.1 is an extension of Schauder-Tychonoff's fixed point theorem. Since our main purpose is a generalization of the continuity condition for set valued mappings, it seems that there is no advantage in our approach for cases with continuous functions. The above corollary suggests, however, that our characterization of mappings under the direction structure seems to exhaust all the essential features in the notion of continuity at least for fixed point arguments.

As an important result for cases with set valued mappings, we show the following corollary. Also in this result, we reformulate our concept into more familiar notion, the existence of (locally definable) continuous selections.

COROLLARY 3.2 : (Fixed Point Theorem for Mappings Having Continuous Local Selections) *Let X be a non-empty compact convex subset of Hausdorff topological vector space E satisfying condition (T), and let $\varphi : X \rightarrow X$ be a non-empty convex valued correspondence. If φ has, locally, a continuous selection at each $x \in X$ such that $x \notin \varphi(x)$, then φ has a fixed point.*

PROOF : Suppose that φ has no fixed point. Then, for each $x \in X$, there is an open neighbourhood U_x of x and a continuous function $f^x : U_x \rightarrow X$ such that $f^x(z) \in \varphi(z)$ for all $z \in U_x$. Since X is compact there are finite covering U_{x^1}, \dots, U_{x^m} and local selections, f^{x^1}, \dots, f^{x^m} . Let $\alpha^t : U_{x^t} \rightarrow [0, 1]$, $t = 1, \dots, m$ be the partition of unity subordinated to the finite covering. Then, $f : X \rightarrow X$

¹⁶A simple example is the space ℓ^p for $0 < p < 1$ under the pseudo ℓ^p -norm, $\|(x_i)_{i=1}^\infty\| = \sum_{i=1}^\infty |x_i|^p$.

defined as $f(x) = \sum_{t=1}^m \alpha^t(x) f^{x^t}(x)$, where $f^{x^t}(x)$ for $x \notin U_{x^t}$ is defined as 0, is a continuous selection of φ since φ is convex valued. Then, by Corollary 3.1, f has a fixed point, so that φ has a fixed point, a contradiction. ■

In the above proof, if φ has a special type of local continuous selections, local selections of constant functions, (e.g., if φ has an open lower section at everywhere), then condition (T) will be omitted. Indeed, for such a case, the range of f is finite dimensional, so that Brouwer's fixed point theorem is sufficient for f to have a fixed point. The situation is completely the same when the range of each local continuous selection is not a single point (constant function) but a subset of finite dimensional space. This may also be considered as a corollary to the case that we have treated in Theorem 1 as the locally common direction. But we shall write it here as the most simple case of the previous corollary.

COROLLARY 3.3 : (A Generalization of Browder's Theorem) *Let X be a non-empty compact convex subset of Hausdorff topological vector space E , and let $\varphi : X \rightarrow X$ be a non-empty convex valued correspondence. If φ has, locally, a continuous selection whose range is a subset of finite dimensional subspace at each $x \in X$ such that $x \notin \varphi(x)$. Then, φ has a fixed point.*

PROOF : As stated above, since the range of each f^{x^t} , $f^{x^t}(U_{x^t})$ in the proof of the previous corollary is in a finite dimensional subspace L_t of E , the range of f is also in a finite dimensional subspace $L = \sum_{t=1}^m L_t$ of E . Hence, the function, f , restricted on $L \cap X$ has a fixed point by Brouwer's fixed point theorem, so that we have a contradiction. ■

These corollaries shows that there may exist a trade-off between the generality for the vector space topology on the base set and the variety of mappings to which we want to show the existence of fixed points. Our approach, however, also suggests that the concept of direction structure brings about a unified view point on these topologies and mappings. For example, compare Corollary 2.2 and Corollary 3.1. In Corollary 3.1, the condition for the direction structure is completely described as a property on the topological vector space, Condition (T). On the contrary, in Corollary 3.1, the condition is described, completely, as a property for the mapping f .

3 THEOREMS ON ECONOMIC EQUILIBRIA

In this section, we apply fixed point theorems in the previous section to several problems in the economic equilibrium theory.

Let $P : X \rightarrow X$ be a (possibly empty valued) correspondence on a subset X of a topological vector space E to itself. Assume that P satisfies

$$(\text{Irreflexivity}) \forall x \in X, x \notin P(x).$$

In the following, we regard X as an individual choice set and $P(x) \subset X$ as the set of points which are preferred to x for each $x \in X$. Then, an element $x^* \in X$ may be interpreted as a *maximal element* for the preference correspondence, P , if $P(x^*) = \emptyset$.

In the previous section, we have seen in Corollary 2.1 that a fixed point theorem may easily be modified to the existence theorem on maximal elements. I.e., the existence of maximal elements

for an irreflexive mapping may be considered as a contrapositive assertion to the existence of fixed point for a non-empty valued correspondence. Since in the maximal element existence problem, mapping P directly represents the individual preferences, the importance of our generalization of mappings for fixed point theorem (in the previous section) should be measured by the generality for a representation of our general preferences.

We emphasize that as a condition for the preference, “better sets have locally similar directions” is not only mathematically general but also intuitively natural. It is far more natural than the continuity. Moreover, there are many concrete examples that may not be treated in the standard argument but may be treated in our scope. For example, there is an important sort of *ordered* preferences that are *not continuous* to fail to have open lower sections, the *lexicographic ordering*. There also exists an important sort of ordered and continuous preferences that are *not complete* to fail to have open lower sections (consider the relation \leq in R^n representing the strict monotonicity at each point). (See Figure 2. In each case, the better set at x is denoted by the shaded area.)

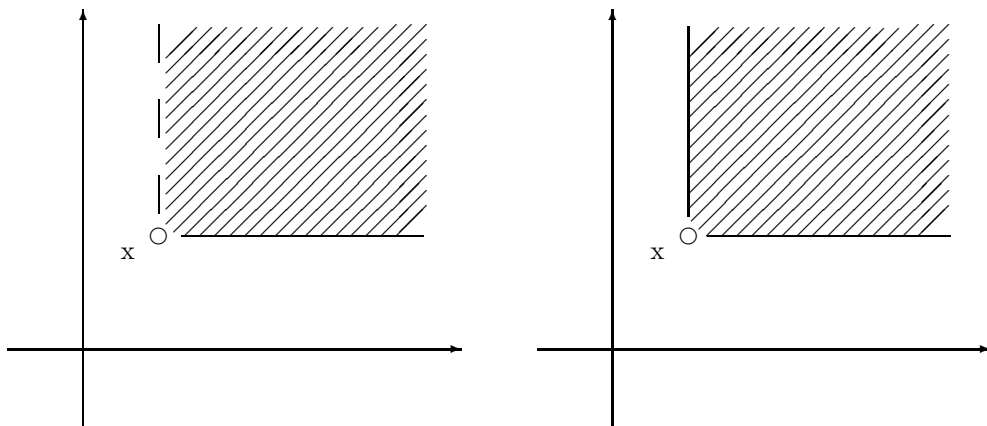


Figure 2: Lexicographic Orderings and Orderings in Vector Spaces

We write here a generalized version of Corollary 2.1, a theorem on the existence of maximal elements, whose assumptions are taken as weak as possible from the economic view point. Preceding the theorem, we describe several basic settings for the economic equilibrium arguments.

(B0) [Basic Settings for the Economic Equilibrium Theory]: Let E be a Hausdorff topological vector space, X be a compact subset of E , and E'_X be a convex subset of the set of linear functionals on E , E^* , having a Hausdorff vector space topology such that the closure D of E'_X in E^* is compact, and the restriction of bilinear form $\langle p, x \rangle$ on $E'_X \times X$ is jointly continuous.¹⁷

We shall use above basic settings, repeatedly, throughout this section.

¹⁷The assumed property for the bilinear form may be satisfied under the standard situation for the commodity-price duality in economic equilibrium theory. For example, given a duality (E, F) , suppose that E has the Mackey topology $\tau(E, F)$ and E'_X is a dense subset of a $\sigma(F, E)$ compact subset D of F . The joint continuity for the bilinear form has an economic meaning that we are considering the situation in which with simultaneous small changes in prices and amounts for commodities associate small changes in their total values.

THEOREM 4 : (Existence of Maximal Elements) *Under (B0), suppose that $P : X \rightarrow X$ is a (possibly empty valued) correspondence such that for each $x \in X$, there is a neighbourhood U_x of x and an upper semicontinuous compact valued correspondence $\theta_x : U_x \rightarrow E'_X$ satisfying that $\langle p, w - z \rangle > 0$ for all $p \in \theta(z)$, $w \in P(z)$, and $z \in U_x$ as long as $P(x) \neq \emptyset$. Then, there is a maximal element $x^* \in X$, $P(x^*) = \emptyset$.*

PROOF : Assume the contrary. Then, for all $x \in X$, there is a neighbourhood U_x of x and an upper semicontinuous correspondence $\theta_x : U_x \rightarrow E'_X$ satisfying that $\langle p, w - z \rangle > 0$ for all $p \in \theta(z)$ and $w \in P(z)$. Since X is compact, there is a finite covering U_{x^1}, \dots, U_{x^m} of X , so that by using the partition of unity $\alpha^t : U_{x^t} \rightarrow [0, 1]$, $t = 1, \dots, m$ subordinated to U_{x^1}, \dots, U_{x^m} , we obtain an upper semicontinuous correspondence $\theta : X \rightarrow E'_X$ as

$$\theta(x) = \sum_{t=1}^m \alpha^t(x) \theta_{x^t}(x),$$

where $\theta_{x^t}(x)$ is defined to be $\{0\}$ for all $x \notin U_{x^t}$. Since $\theta : X \rightarrow E'_X$ is upper semicontinuous and compact valued, and since the bilinear form $\langle \cdot, \cdot \rangle$ on $X \times E'_X$ is jointly continuous, the direction structure $V(x, y)$ on X defined by $V(x, y) = \{w \in X \mid \langle p, w - x \rangle > 0\}$ if $y \in \{w \in X \mid \langle p, w - x \rangle > 0\}$, else $V(x, y) = \emptyset$, is lower topological and P has a locally fixed direction at each $x \in X$ under V . Hence, by Theorem 2, P has a fixed point. Since $\langle p, w - z \rangle > 0$ for all $p \in \theta(z)$, $w \in P(z)$, and $z \in U_x$ as long as $P(x) \neq \emptyset$, P cannot have a fixed point, so that we have a contradiction. ■

We next consider the problem on the existence of maximal elements among many agents, i.e., the social optima. Adding to (B0), we use the following settings:

(B1) [Consumers and Producers]: Let X_1, \dots, X_m and Y_1, \dots, Y_n be compact convex subsets of X . Moreover, let $\omega^1, \dots, \omega^m$ are points in E such that $\sum_{i=1}^m X_i \cap (\sum_{j=1}^n Y_j + \sum_{i=1}^m \omega^i) \neq \emptyset$.

We call the set $\mathbf{Alloc} = \prod_{i=1}^m X_i$ the *set of allocations* (for consumers) and the set $\mathbf{Falloc} = \{(x^i)_{i=1}^m \in \mathbf{Alloc} \mid \sum_{i=1}^m x^i = \sum_{j=1}^n y^j + \sum_{i=1}^m \omega^i, (y^j)_{j=1}^n \in \prod_{j=1}^n Y_j\}$ the *set of feasible allocations*. Under (B1), we see that \mathbf{Falloc} is a non-empty compact convex subset of \mathbf{Alloc} .

Together with (B0) and (B1), let us assume that for each $i = 1, \dots, m$, there are two correspondences $P_i : \mathbf{Alloc} \rightarrow X_i$ and $\tilde{P}_i : \mathbf{Alloc} \rightarrow X_i$, the *strong* and *weak preference correspondences*, respectively, satisfying that $\forall x = (x^1, \dots, x^m) \in \mathbf{Alloc}$, $x^i \notin P_i(x)$ (the irreflexivity), $x^i \in \tilde{P}_i(x)$ (the reflexivity), and $P_i(x) \subset \tilde{P}_i(x)$. Then, let us define a correspondence, $P : \mathbf{Alloc} \rightarrow \mathbf{Alloc}$, as follows:

$$P(x) = \{w = (w^1, \dots, w^m) \mid \forall i, w^i \in \tilde{P}_i(x) \text{ and } \exists i, w^i \in P_i(x)\}$$

For allocations $x, y \in \mathbf{Alloc}$, y is said to be *Pareto superior* to x if and only if the condition $y \in P(x)$ is satisfied. It is easy to check that for all $z \in \mathbf{Alloc}$, $z \notin P(z)$, i.e., $P : \mathbf{Alloc} \rightarrow \mathbf{Alloc}$ is an irreflexive correspondence. A feasible allocation $x \in \mathbf{Falloc}$ is said to be *Pareto optimal* if $P(x) \cap \mathbf{Falloc} = \emptyset$. The next theorem shows that the existence of Pareto optimal allocations may be assured even for cases with non-ordered, non-convex, and non-continuous individual preferences. Note that the necessary condition for individual preferences to assure the existence of social optima (Pareto optimal allocations), (D), is essentially the same as the necessary condition for the existence of individual optima (maximal elements) in Theorem 4.

THEOREM 5 : (Existence of Pareto Optimal Allocations) *Assume that for each $i = 1, \dots, m$, P_i and \tilde{P}_i satisfies the following conditions for directions of mappings.*

(D) *For each $x \in \mathbf{Alloc}$, (resp., for each $x \in \mathbf{Alloc}$ such that $P_i(x) \neq \emptyset$), there are a neighbourhood U_x of x and an upper semicontinuous compact valued correspondence $\theta_x^i : U_x \rightarrow E'_X$ such that $\langle p, w_i - z_i \rangle \geq 0$ (resp., $\langle p, w_i - z_i \rangle > 0$) for all $p \in \theta_x^i(z)$, $w_i \in \tilde{P}_i(z)$, (resp., $w_i \in P_i(z)$), and $z = (z_1, \dots, z_m) \in U_x$.*

Then, there is a Pareto optimal allocation.

PROOF : Assume the contrary. Then, the correspondence $P_F : \mathbf{Falloc} \ni x \mapsto P(x) \cap \mathbf{Falloc}$ is non-empty valued. By condition (D) and the compactness of \mathbf{Alloc} , through the argument using the partition of unity, we obtain an upper semicontinuous compact valued correspondence $\theta^i : \mathbf{Alloc} \rightarrow E'_X$ for each i such that $\forall z \in \mathbf{Alloc}$ and $\forall p \in \theta^i(z)$, $(w_i \in \tilde{P}_i(z)) \rightarrow (\langle p, w_i - z_i \rangle \geq 0)$ and $(w_i \in P_i(z)) \rightarrow (\langle p, w_i - z_i \rangle > 0)$. For each i and $z \in \mathbf{Alloc}$, denote by $\hat{P}_i(z)$, (resp., $\hat{\tilde{P}}_i(z)$), the set $\{w_i \in X_i | \forall p \in \theta^i(z), \langle p, w_i - z_i \rangle > 0\}$, (resp., the set $\{w_i \in X_i | \forall p \in \theta^i(z), \langle p, w_i - z_i \rangle \geq 0\}$). Clearly, for each i and $z \in \mathbf{Alloc}$, $\hat{P}_i(z) \supset P_i(z)$ and $\hat{\tilde{P}}_i(z) \supset \tilde{P}_i(z)$. Since $\hat{P}_i(z)$ is open, and since the sum operation is continuous, the non-emptiness of P_F means that for each $w \in \mathbf{Falloc}$, there is an neighbourhood O_w in \mathbf{Falloc} and an index of consumer $i(w)$ such that for each $z \in O_w$, there is an element $y = (y_1, \dots, y_m) \in \mathbf{Falloc}$ such that $y_{i(w)} \in \hat{P}_{i(w)}(z)$. Since \mathbf{Falloc} is compact, there is a finite covering O_{w^1}, \dots, O_{w^k} of \mathbf{Falloc} and the partition of unity, $\alpha_t : O_{w^t} \rightarrow [0, 1]$, $t = 1, \dots, k$, subordinated to it. Let us define a correspondence on \mathbf{Falloc} to \mathbf{Falloc} , Φ as, for each $z = (z_1, \dots, z_m) \in \mathbf{Falloc}$,

$$\begin{aligned} \Phi(z) &= z + \alpha^t(z)(\Phi(z)^t - z), \text{ where} \\ \Phi^t(z) &= \hat{\tilde{P}}_1(z) \times \dots \times \hat{\tilde{P}}_{i(w^t)-1}(z) \times \hat{P}_{i(w^t)}(z) \times \hat{\tilde{P}}_{i(w^t)+1}(z) \times \dots \times \hat{\tilde{P}}_m(z) \cap \mathbf{Falloc}. \end{aligned}$$

Φ is convex valued since each Φ^t is. Φ has non-empty valued since each Φ^t is non-empty valued as long as P_F is. Moreover, by defining $\theta(z) : \mathbf{Alloc} \rightarrow \mathbf{Alloc}$ and $\Psi : \mathbf{Falloc} \rightarrow \mathbf{Falloc}$, respectively, as

$$\begin{aligned} \theta(z) &= (\theta^1(z), \dots, \theta^m(z)), \text{ and} \\ \Psi(z) &= \{w \in \mathbf{Falloc} | \forall p \in \theta(z), \langle p, w \rangle > 0\}, \end{aligned}$$

we have $\Phi(z) \subset \Psi(z)$, so that the correspondence, Ψ , is a non-empty convex valued on \mathbf{Falloc} to \mathbf{Falloc} having no fixed point. Note that θ is also compact valued upper semicontinuous correspondence on \mathbf{Alloc} to $E'_X{}^{(m)}$, where $E'_X{}^{(m)}$ denotes the m -th product of E'_X . Then, for all $w^* \in \Psi(z^*)$, there is an $\epsilon^* > 0$ such that $\langle p, w^* \rangle > \epsilon^* > 0$ by the compactness of $\theta(z^*)$, hence, by the joint continuity of the duality operation, (precisely, for each component,) there is a neighbourhood O^* of z^* such that for all $z \in O^*$, $w^* \in \Psi(z)$. Therefore, Ψ has a locally continuous (fixed) selection at each z , so that by Corollary 3.3, Ψ has a fixed point and we have a contradiction. ■

The set E'_X may be interpreted as the set of *prices*. For each price $p \in E'_X$, we may define a non-empty set of *excess demands*, $\zeta(p) \subset X$, as the consequence of agents' maximization behaviors.¹⁸

¹⁸Under (B0) and (B1), if we suppose the ordinary private ownership structure as in Debreu (1959), Theorem 4 clearly assures the non-emptiness for each agent's optimal behaviors, $\zeta(p)$. Conditions (B2) and (B3) below may also be satisfied. For condition (E) in the next theorem, however, we do not have such an individual preference foundation. In this sense, it is more natural for us to consider the excess demand correspondence, $p \mapsto \zeta(p)$, as a primitive notion (the excess demand approach). Needless to say, condition (E) is more general than to assume the ordinary situation such that the excess demand correspondence is compact convex valued and upper semi-continuous.

If we add the following conditions for the excess demand correspondence, ζ , we may argue for the market equilibrium and the existence of equilibrium prices.

(B2) [Walras' Law]: For each $p \in E'_X$, $\langle p, z \rangle = 0$ for all $z \in \zeta(p)$.

(B3) If $0 \notin \zeta(p)$, there exists $q \in E'_X$ such that $\langle q, z \rangle > 0$ for all $z \in \zeta(p)$.

Condition (B2) is always satisfied as long as every consumer satisfies their ordinary budget constraint with equality. Condition (B3) says that the price set is taken to be sufficiently large so that we may adjust any non-zero amount of excess demands. We say that the price p is *adjusted* by price q if the relation in (B3), $\langle q, z \rangle > 0$ for all $z \in \zeta(p)$, is satisfied. By (B2), p is not adjusted by p for all $p \in E'_X$. A price, $p^* \in E'_X$ is said to be an *equilibrium price* if $0 \in \zeta(p^*)$. Under (B3), a price p^* is an equilibrium price if and only if p^* is not adjusted by q for all $q \in E'_X$. The mathematical result on the existence of equilibrium for this type of abstract settings is known as Gale-Nikaido-Debreu theorem.¹⁹ In our general settings, the theorem may be extended as follows.

THEOREM 6 : (Existence of Price Equilibrium) *Assume (B0), and let us consider an excess demand correspondence, $\zeta : E'_X \rightarrow X$, satisfying (B2) and (B3). Moreover, suppose that on the closure D of E'_X , condition (T) in Corollary 3.1 is satisfied, and that ζ has a continuous local direction at each non-equilibrium point in the following sense.*

(E) *For each r in D , if r is not an equilibrium price, there is a neighbourhood U_r of r in D and a continuous function $q_r : U_r \rightarrow E'_X$ such that for all $p \in U_r \cap E'_X$, $\langle q_r, z \rangle > 0$ for all $z \in \zeta(p)$.*

Then, there is an equilibrium price $p^ \in E'_X$, $0 \in \zeta(p^*)$.*

PROOF : Assume the contrary, so that there is no $p \in E'_X$ such that $0 \in \zeta(p)$. (Note that the closure, D , of E'_X in E^* is compact and convex.) Then for all r in D , there is a neighbourhood U_r of r in D and a continuous function $q_r : U_r \rightarrow E'_X$ satisfying the condition stated above. Since the closure of E'_X is compact, there are finite subcovering U_{r_1}, \dots, U_{r_k} of D . Let $\alpha^t : U_{r_t} \rightarrow X$, $t = 1, \dots, k$ be the partition of unity subordinated to U_{r_t} , $t = 1, \dots, k$. Then, the mapping $f : D \ni r \mapsto \sum_{t=1}^k \alpha^t(r) q_{r_t}(r) \in D$ is continuous and has no fixed point since for all t such that $p \in U_{r_t}$, $\langle q_{r_t}, z \rangle > 0$ for all $z \in \zeta(p)$ though $\langle p, z \rangle$ should be 0 under (B2). Since condition (T) on D is satisfied, f has a fixed point by Corollary 3.1, so that we have a contradiction. ■

In an extension of Gale-Nikaido-Debreu theorem, condition (E) may be considered as one of the weakest requirements for the continuity of excess demand correspondences. Except for directions of mappings, there is no topological requirement for the set of values at each point.²⁰ It should also be noted, however, that condition (E) includes the so called *boundary condition* for the excess demand correspondence (on the boundary $D \setminus E'_X$ of E'_X in E^*). As a boundary condition, (i.e., as a condition for points $r \in D \setminus E'_X$), condition (E) may not be called the weakest one.²¹ There is an

¹⁹See, for example, Debreu (1956). For one of the most general treatments in locally convex space with acyclic valued correspondences, see Nikaido (1957) and Nikaido (1959; Section 5.2).

²⁰Of course, it is easy to check that every non-empty closed convex valued upper semicontinuous correspondence satisfies condition (E).

²¹In condition (E), the boundary behavior is described for a neighbourhood of each boundary point. Boundary conditions of the weakest type usually treat each boundary point independently. See, for example, Aliprantis and Brown (1983), Mehta and Tarafdar (1987), etc. For a more general and unified treatment, see Urai (2000; Section 4).

alternative way for assuring the existence of equilibrium prices under more general conditions for boundary points, though requirements for the continuity in (E) should be strengthened.

If we write $B(p) = \{q \in E'_X \mid \exists z \in \zeta(q), \langle p, z \rangle \leq 0\}$ for each $p \in E'_X$, (the set of price q which may not be adjusted by price p), then the following (E') (a slight modification of the continuity condition in (E)) is a sufficient condition for $B(p)$ to be a closed subset of E'_X .

(E') For each q in E'_X , if q is not an equilibrium price, there is a neighbourhood U_q of q in E'_X and a point $p_q \in E'_X$ such that for all $q' \in U_q$, $\langle p_q, z \rangle > 0$ for all $z \in \zeta(q')$.

It is also easy to check that for each finite set p^1, \dots, p^k , $\text{co}\{p^1, \dots, p^k\} \subset \bigcup_{t=1}^k B(p^t)$. Hence, by using Knaster-Kuratowski-Mazurkiewicz theorem together with the following boundary condition, we also obtain the existence of equilibrium prices.

(B4) [Boundary Condition]: For all r in the boundary, $\partial E'_X$, of $E'_X \subset D$ and for all net $\{p^\nu, \nu \in \mathcal{N}\}$ in E'_X converging to r , there are an element $q_r \in E'_X$ and a subnet $\{p^{\nu(\mu)}, \mu \in \mathcal{M}\}$ of $\{p^\nu, \nu \in \mathcal{N}\}$ such that $\langle q_r, z \rangle > 0$ for all $z \in \zeta(p^{\nu(\mu)})$ for all $\mu \in \mathcal{M}$.

In this case, the generality of the continuity condition (E') together with the boundary condition (B4) is essentially the same with the condition used in the market equilibrium existence theorem in Urai (2000; Theorem 8).

It is also possible to apply theorems in the previous section and settings (B1), (B2) to the existence of Nash equilibrium and Generalized Nash equilibrium for an abstract economy of the Arrow-Debreu type. In such cases, we obtain various results on the existence of equilibrium with (possibly) non-ordered, non-convex, and/or non-continuous preferences and constraint correspondences in Hausdorff topological vector spaces. For some results, see Urai and Yoshimachi (2002).

REFERENCES

- Aliprantis, C. D. and Brown, D. J. (1983): "Equilibria in markets with a Riesz space of commodities," *Journal of Mathematical Economics* 11, 189–207.
- Arrow, K. J. and Debreu, G. (1954): "Existence of an equilibrium for a competitive economy," *Econometrica* 22, 265–290.
- Browder, F. (1968): "The fixed point theory of multi-valued mappings in topological vector spaces," *Mathematical Annals* 177, 283–301.
- Debreu, G. (1956): "Market equilibrium," *Proceedings of the National Academy of Sciences of the U.S.A.* 42, 876–878. Reprinted as Chapter 7 in G. Debreu, *Mathematical Economics*, Cambridge University Press, Cambridge, 1983.
- Debreu, G. (1959): *Theory of Value*. Yale University Press, New Haven, CT.
- Gale, D. and Mas-Colell, A. (1975): "An equilibrium existence theorem for a general model without ordered preferences," *Journal of Mathematical Economics* 2, 9–15. (For some corrections see *Journal of Mathematical Economics*, vol.6: 297–298, 1979.).
- Mehta, G. and Tarafdar, E. (1987): "Infinite-Dimensional Gale-Nikaido-Debreu theorem and a Fixed-point Theorem of Tarafdar," *Journal of Economic Theory* 41, 333–339.

- Nikaido, H. (1956): "On the classical multilateral exchange problem," *Metroeconomica* 8, 135–145.
A supplementary note: *Metroeconomica*, 8, 209–210, 1957.
- Nikaido, H. (1957): "Existence of equilibrium based on the Walras' law," ISER Discussion Paper No. 2, Institution of Social and Economic Research.
- Nikaido, H. (1959): "Coincidence and some systems of inequalities," *Journal of The Mathematical Society of Japan* 11(4), 354–373.
- Schaefer, H. H. (1971): *Topological Vector Spaces*. Springer-Verlag, New York/Berlin.
- Shafer, W. and H.F.Sonnenschein, (1975): "Equilibrium in abstract economies without ordered preferences," *Journal of Mathematical Economics* 2, 345–348.
- Urai, K. (2000): "Fixed point theorems and the existence of economic equilibria based on conditions for local directions of mappings," *Advances in Mathematical Economics* 2, 87–118.
- Urai, K. and Hayashi, T. (2000): "A generalization of continuity and convexity conditions for correspondences in the economic equilibrium theory," *The Japanese Economic Review* 51(4), 583–595.
- Urai, K. and Yoshimachi, A. (2002): "Existence of equilibrium with non-convex constraint correspondences: Including an application for the default economy," Discussion Paper No. 02-06, Faculty of Economics and Osaka School of International Public Policy, Osaka University.